ISE 315: Engineering Statistics

Lecture 6: Statistical Intervals for a Single Sample (Part 2)

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Based on Montgomery & Runger, Applied Statistics and Probability for Engineers, 6th Ed.

Lecture 6

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Lecture 6 Outline:

- Announcements:
  - HW #2 due next week
  - Major Exam 1 in Week 8 (Chapters 7, 8, 9, 10)
  - Quiz #2 next Wednesday (up to 5 questions about Chapter 8)
- Z Table and Problem Examples
- CI for Mean (σ^2 unknown) , Variance, and Proportion

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
1.6	0.9452	0.9463	0.9474	$\cdots$	0.9505	0.9515	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
1.9	0.9713	0.9719	0.9726	$\cdots$	0.9744	0.9750	$\cdots$
2.0	0.9772	0.9778	0.9783	$\cdots$	0.9798	0.9803	$\cdots$
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Example: $\Phi(1.96) = P(Z \leq 1.96) = 0.9750$

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1.9	0.9713	$\cdots$	0.9744	**0.9750**	0.9756	$\cdots$
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Common Critical Values

Conf. Level	α	Two-sided (CI)		One-sided	
		$\Phi(z_{\alpha/2})$	$z_{\alpha/2}$	$\Phi(z_\alpha)$	z_α
90%	0.10	0.95	1.645	0.90	1.282
95%	0.05	0.975	1.960	0.95	1.645
99%	0.01	0.995	2.576	0.99	2.326

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Given:

- $n = 25$, $\bar{x} = 0.2731$, $\sigma = 0.02$
- Confidence level: 95% $\Rightarrow \alpha = 0.05 \Rightarrow z_{0.025} = 1.96$

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$$\text{CI: } \bar{x} \pm E = 0.2731 \pm 0.00784 = \boxed{(0.2653, 0.2809) \text{ inches}}$$

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$$= \left(\frac{117.6}{15}\right)^2 = (7.84)^2 = 61.47 \approx 62 \text{ (round up!)}$$

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$$= 1850 - 1.645 \times 25 = 1850 - 41.125 = \boxed{1808.9 \text{ kg}}$$

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4. **Sample Proportion:**
   - $\hat{p} = X/n$ where $X =$ successes
   - Estimates proportion p
   - Uses Normal approximation (if some conditions are met)

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As df $\to \infty$: $t \to z$ and $\chi^2/df \to 1$

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When in doubt about assumptions, use larger sample sizes!

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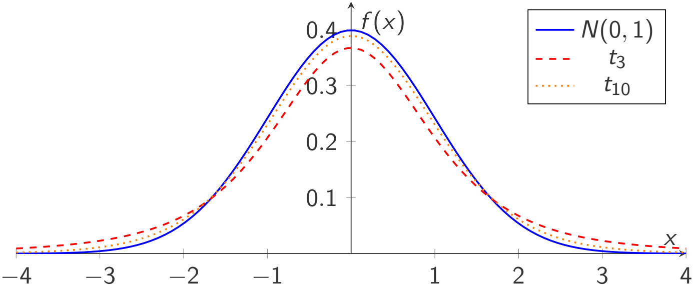
- As $n \to \infty$: $t_{n-1} \to N(0,1)$

Comparing t and Normal Distributions



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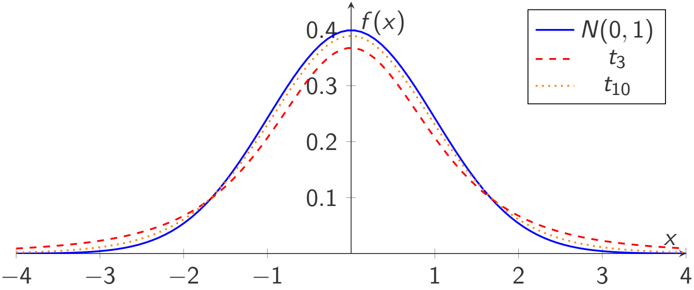

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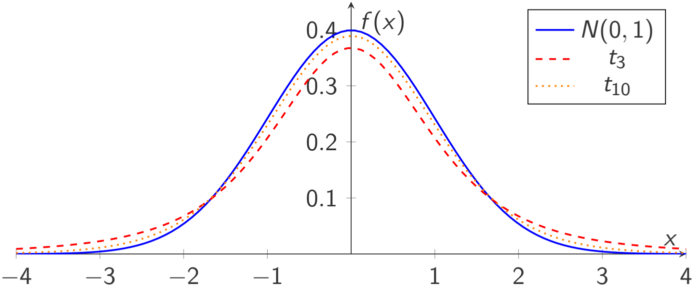
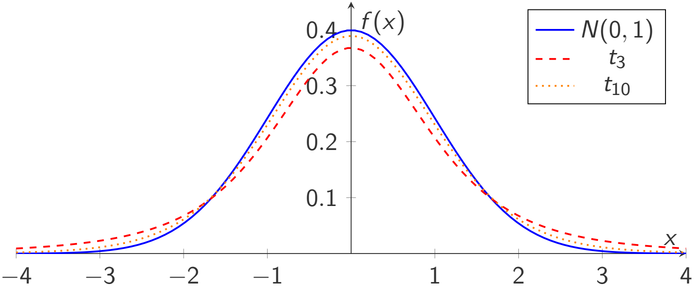


Comparing t and Normal Distributions



Key insight: Lower df ⇒ heavier tails ⇒ larger critical values ⇒ wider CI

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Sample Size for Proportion: $n = \left(\frac{z_{\alpha/2}}{E}\right)^2 \hat{p}(1-\hat{p})$ (use $\hat{p} = 0.5$ if unknown)

t-Distribution Critical Values ($t_{\alpha,\nu}$ = value where $P(T > t_{\alpha,\nu}) = \alpha$)

	Upper-tail probability α			
df (ν)	**0.10**	**0.05**	**0.025**	**0.01**
1	3.078	6.314	12.706	31.821
2	1.886	2.920	4.303	6.965
3	1.638	2.353	3.182	4.541
5	1.476	2.015	2.571	3.365
10	1.372	1.812	2.228	2.764
15	1.341	1.753	2.131	2.602
20	1.325	1.725	2.086	2.528
30	1.310	1.697	2.042	2.457
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Example: For 95% two-sided CI with $n = 11$ samples (df $= 10$): Use $t_{0.025,10} = 2.228$

χ^2-Distribution Critical Values ($\chi^2_{\alpha,\nu}$ = value s.t. $P(\chi^2 > \chi^2_{\alpha,\nu}) = \alpha$)

	Upper-tail probability α			
df (ν)	**0.975**	**0.95**	**0.05**	**0.025**
1	0.001	0.004	3.841	5.024
2	0.051	0.103	5.991	7.378
5	0.831	1.145	11.070	12.833
9	2.700	3.325	16.919	19.023
10	3.247	3.940	18.307	20.483
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Example: For 95% CI for σ^2 with $n = 10$ ($df = 9$): Use $\chi^2_{0.975,9} = 2.700$ and $\chi^2_{0.025,9} = 19.023$

Summary

- When σ is **unknown**, use the t-**distribution**:
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- For **proportion** p, use **Normal approximation**:
  - Requires large sample: $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$
  - Standard error: $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

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