ISE 315: Engineering Statistics

Lecture 11: Statistical Inference for Two Samples

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Industrial and Systems Engineering, KFUPM

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Based on Montgomery & Runger, Applied Statistics and Probability for Engineers, 6th Ed.

Lecture 11

Statistical Inference for Two Samples (Chapter 10)

Announcements

- Mid-Semester Course Eval: Only 4 so far. Please fill it out

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- **Major 1: Week 8, Thursday March 5th (80 minutes)**

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- **3 Practice Problems will be posted as your Major 1 rehearsal.**
  Complete all by Tue, Feb 24 $\Rightarrow$ skip HW 5 (score $=$ the highest)

Upcoming Schedule

Lecture	Date	Topic / Activity
11	Sun, Feb 15	Ch. 10: Two-Sample Inference (Part 1)

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HW 5 required only if you do not complete all the long practice quizzes

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	Holiday Break	**No class**

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- Worked examples with engineering context
- Practice problems for Quiz 3 (15 minutes)

The α–β Tradeoff and OC Curves

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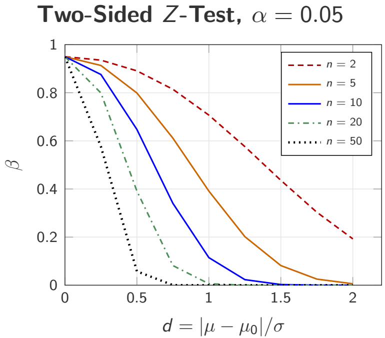
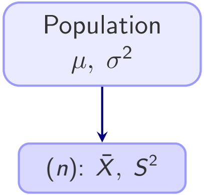


From One Sample to Two Samples



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$H_0: \mu = \mu_0$ or $H_0: \sigma^2 = \sigma_0^2$

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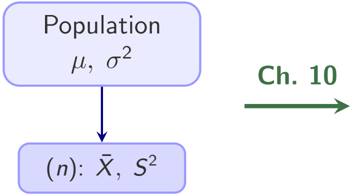

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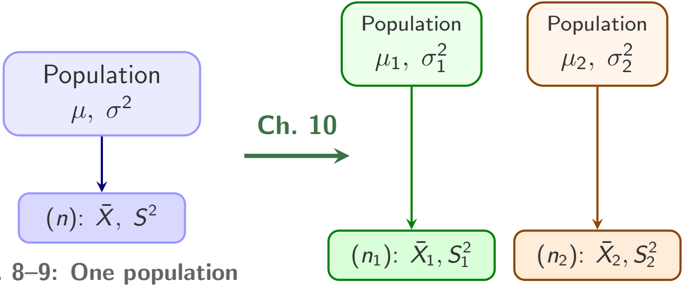

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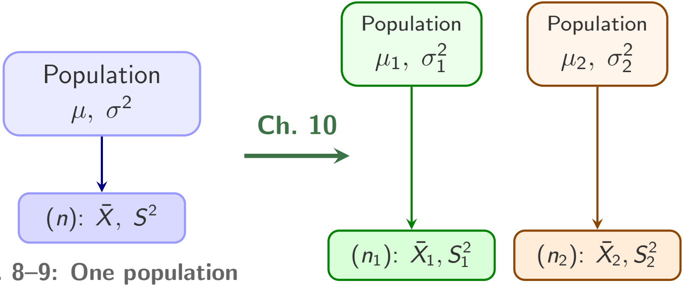

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From Lecture 4:

$E[\bar{X}_1 - \bar{X}_2] = \mu_1 - \mu_2$

$\mathrm{Var}[\bar{X}_1 - \bar{X}_2] = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

Setting Up the Two-Sample Problem

Difference in Means, σ_1^2 and σ_2^2 Known

Setup: Two independent random samples:

- Sample 1: n_1 observations from population with mean μ_1, known variance σ_1^2
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This follows from the fact that the sum (or difference) of independent normal random variables is also normal.

Confidence Interval for $\mu_1 - \mu_2$ (σ_1^2, σ_2^2 Known)

A $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$:

$$(\bar{x}_1 - \bar{x}_2) \ \pm \ z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

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Compare to the single-sample case:

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Can you summarize the key differences?

Hypothesis Test for $\mu_1 - \mu_2$ (σ_1^2, σ_2^2 Known)

Hypotheses (testing whether the difference equals some value Δ_0, often 0):

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \quad \text{vs} \quad H_1 : \mu_1 - \mu_2 \neq \Delta_0$$

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Under H_0, $Z_0 \sim N(0, 1)$.

Decision rules: Reject if

- **test statistic is further in the tail than the critical** or **P-value** $\leq \alpha$.

Decision Rules for Two-Sample Z-Test

Alternative H_1	**Reject** H_0 **if**	**P-value**
$\mu_1 - \mu_2 \neq \Delta_0$	$\|Z_0\| > z_{\alpha/2}$	$2\left[1 - \Phi(\|z_0\|)\right]$
$\mu_1 - \mu_2 > \Delta_0$	$Z_0 > z_\alpha$	$1 - \Phi(z_0)$
$\mu_1 - \mu_2 < \Delta_0$	$Z_0 < -z_\alpha$	$\Phi(z_0)$

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Note: This is structurally identical to the single-sample Z-test (only differs in how we compute Z_0).

Example: Comparing Two Drilling Fluids

A petroleum engineer wants to compare the corrosion rates (mm/year) of two drilling fluids on steel pipe.

- Fluid A: $n_1 = 15$, $\bar{x}_1 = 4.85$, $\sigma_1 = 0.60$ (known)
- Fluid B: $n_2 = 17$, $\bar{x}_2 = 5.16$, $\sigma_2 = 0.75$ (known)

Is there evidence that the mean corrosion rates differ? Use $\alpha = 0.05$.

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Step 2: $H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 - \mu_2 \neq 0$, $\alpha = 0.05$.

Example: Drilling Fluids (Solution)

Step 3: Test statistic:

$$Z_0 = \frac{(4.85 - 5.16) - 0}{\sqrt{\frac{0.60^2}{15} + \frac{0.75^2}{17}}} = \frac{-0.31}{\sqrt{0.024 + 0.0331}} = \frac{-0.31}{0.2390} = -1.297$$

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Step 5: $|Z_0| = 1.297 < 1.96$. Z_0 is **not** in the rejection region.
P-value $= 2[1 - \Phi(1.297)] = 2(1 - 0.9027) = 2(0.0973) = 0.1946$

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Step 6: **Fail to reject H_0.** At $\alpha = 0.05$, there is not sufficient evidence to conclude that the mean corrosion rates of the two drilling fluids differ.

Two Cases When Variances Are Unknown

Difference in Means, σ_1^2 and σ_2^2 Unknown

When σ_1^2 and σ_2^2 are unknown, we estimate them with s_1^2 and s_2^2.

But we need to decide: can we assume $\sigma_1^2 = \sigma_2^2$?

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We "pool" the two sample variances into one combined estimate.
This is called the **pooled t-test**.

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Case 2: $\sigma_1^2 \neq \sigma_2^2$ (unequal variances)
We keep the variances separate and use an **approximate degrees of freedom**.
This is called **Welch's t-test** (Section 10-2.2, next lecture).

The Pooled t-Test (assuming $\sigma_1^2 = \sigma_2^2$)

Pooled variance estimator (weighted-average variance):

$$S_p^2 = \frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}$$

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- **Under H_0, $T_0 \sim t_{n_1+n_2-2}$.**
- **The degrees of freedom is** $n_1 + n_2 - 2$ (because we used both samples to estimate the common variance).

Confidence Interval: Pooled t ($\sigma_1^2 = \sigma_2^2$)

A $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$ when $\sigma_1^2 = \sigma_2^2$:

$$(\bar{x}_1 - \bar{x}_2) \ \pm \ t_{\alpha/2,\, n_1+n_2-2} \ \cdot \ s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

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Compare to single-sample t-CI:

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Compare to single-sample t-CI:

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- Same structure as single-sample t-CI: estimate $\pm$ (critical value)

Confidence Interval: Pooled t ($\sigma_1^2 = \sigma_2^2$)

A $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$ when $\sigma_1^2 = \sigma_2^2$:

$$(\bar{x}_1 - \bar{x}_2) \;\pm\; t_{\alpha/2,\, n_1+n_2-2} \;\cdot\; s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Compare to single-sample t-CI:

$$\bar{x} \;\pm\; t_{\alpha/2,\, n-1} \;\cdot\; \frac{s}{\sqrt{n}}$$

- Same structure as single-sample t-CI: estimate $\pm$ (critical value)
- The standard error now accounts for *two* sample sizes and a *pooled* variance

Decision Rules for Pooled t-Test

Let $\nu = n_1 + n_2 - 2$.

Alternative H_1	Reject H_0 if	P-value bounds
$\mu_1 - \mu_2 \neq \Delta_0$	$\lvert T_0 \rvert > t_{\alpha/2,\,\nu}$	$2\,P(T > \lvert t_0 \rvert)$
$\mu_1 - \mu_2 > \Delta_0$	$T_0 > t_{\alpha,\,\nu}$	$P(T > t_0)$
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Reminder: You can bound P-values from the t-table, or use software for exact values.

Example: Production Line Comparison

Two production lines at a refinery produce polymer pellets. Quality control wants to know if the mean tensile strength (MPa) differs between the two lines. Assume equal population variances.

- Line 1: $n_1 = 12$, $\bar{x}_1 = 22.7$, $s_1 = 1.8$
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Example: Production Lines (Solution)

Step 3: First compute the pooled variance:

$$S_p^2 = \frac{(12-1)(1.8)^2 + (10-1)(2.1)^2}{12+10-2} = \frac{11(3.24) + 9(4.41)}{20}$$
$$= \frac{35.64 + 39.69}{20} = \frac{75.33}{20} = 3.767$$

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95% CI: $1.4 \pm 2.086(0.8312) = 1.4 \pm 1.734 = (-0.334,\ 3.134)$
The interval contains 0, consistent with failing to reject.

Which Test to Use?

Parameter	Conditions	Test Statistic	Distribution
Single sample (Ch. 8–9):			
μ	σ known	Z_0	$N(0, 1)$
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σ^2	Normal pop.	χ_0^2	χ_{n-1}^2
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$\mu_1 - \mu_2$	$\sigma_1^2 \neq \sigma_2^2$ unknown	T_0 (Welch)	*Lecture 12*
μ_D (paired)	Paired data	T_0	*Lecture 12*

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- **Using a two-sample test for paired data.**
  If observations are naturally paired, use the paired t-test (Lecture 12).

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- **Next lecture:** Welch's t-test (unequal variances) and the paired t-test.

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