ISE 315: Engineering Statistics

Lecture 6: Statistical Intervals for a Single Sample (Part 2)

Instructor: Mansur M. Arief, PhD
Industrial and Systems Engineering, KFUPM

Office: 22-219 — Email: mansur.arief@kfupm.edu.sa

Based on Montgomery & Runger, Applied Statistics and Probability for Engineers, 6th Ed.

Lecture 6

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Lecture 6 Outline (Textbook Chapter 3)

- Announcements:
  - HW #2 due next week
  - Major Exam 1 in Week 8 (Textbook Chapters 2–5)
  - Quiz #2 next Wednesday (up to 5 questions about Chapter 3)
- Z Table and Problem Examples
- CI for Mean (σ^2 unknown), Variance, and Proportion

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z	**0.00**	**0.01**	**0.02**	$\cdots$	**0.05**	**0.06**	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
1.6	0.9452	0.9463	0.9474	$\cdots$	0.9505	0.9515	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
1.9	0.9713	0.9719	0.9726	$\cdots$	0.9744	0.9750	$\cdots$
2.0	0.9772	0.9778	0.9783	$\cdots$	0.9798	0.9803	$\cdots$
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Example: $\Phi(1.96) = P(Z \leq 1.96) = 0.9750$

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1.8	0.9641	$\cdots$	0.9678	0.9686	0.9693	$\cdots$
1.9	0.9713	$\cdots$	0.9744	**0.9750**	0.9756	$\cdots$
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Common Critical Values (Table 3.1 in the textbook)

Conf. Level	α	**Two-sided (CI)**		**One-sided**	
		$\Phi(z_{\alpha/2})$	$z_{\alpha/2}$	$\Phi(z_\alpha)$	z_α
90%	0.10	0.95	1.645	0.90	1.282
95%	0.05	0.975	1.960	0.95	1.645
99%	0.01	0.995	2.576	0.99	2.326

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Plan: Two-sided z-interval, (3.1) via Procedure 3.1

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$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{1.96 \times 12}{3}\right)^2$$

$$= \left(\frac{23.52}{3}\right)^2 = (7.84)^2 = 61.47 \approx 62 \text{ users (round up!)}$$

An AV safety team measures the pedestrian-detection distance of a perception model in $n = 16$ test runs. Sample mean $\bar{x} = 185.0$ m, known $\sigma = 10$ m. Find a 95% lower confidence bound for the true mean detection distance.

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$$= 185.0 - 1.645 \times 2.5 = 185.0 - 4.113 = \boxed{180.9 \text{ m}}$$

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4. **Sample Proportion:**
   - $\hat{p} = X/n$ where $X =$ successes
   - Estimates proportion p
   - Uses Normal approximation (if some conditions are met)

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As df $\to \infty$: $t \to z$ and $\chi^2/df \to 1$

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When in doubt about assumptions, use larger sample sizes!

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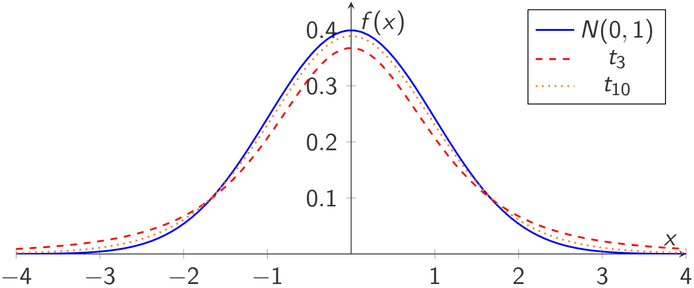
- As $n \to \infty$: $t_{n-1} \to N(0,1)$

Comparing t and Normal Distributions



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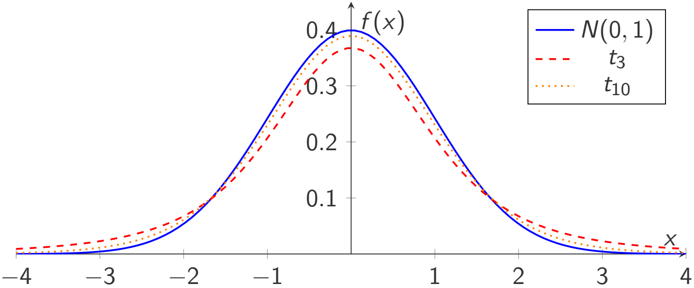

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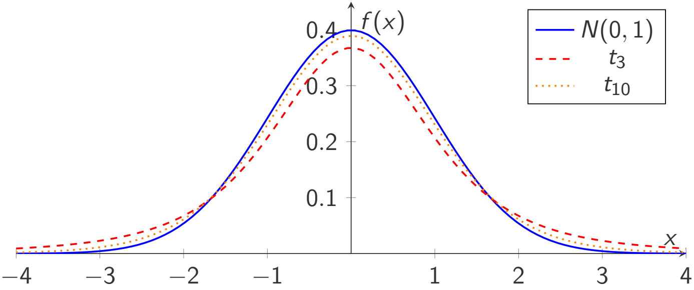

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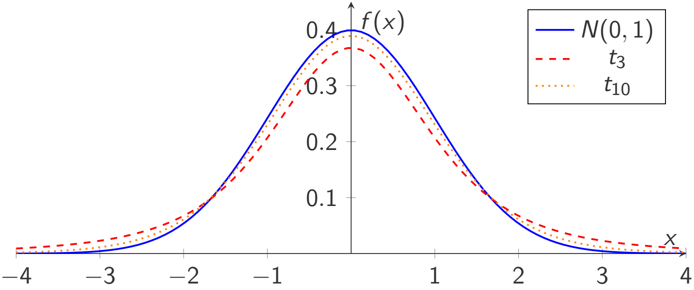

Key point: Lower df $\Rightarrow$ heavier tails $\Rightarrow$ larger critical values $\Rightarrow$ wider CI

CI Formulas: Mean (σ unknown) and Variance

CI for Mean μ (σ unknown) — the t-interval, Procedure 3.2:

$$\bar{x} - t_{\alpha/2,\,n-1}\frac{s}{\sqrt{n}} \;\leq\; \mu \;\leq\; \bar{x} + t_{\alpha/2,\,n-1}\frac{s}{\sqrt{n}} \tag{3.5}$$

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CI for Variance σ^2 — the χ^2-interval, Procedure 3.3:

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For a CI on σ, take square roots of both endpoints of (3.6).

CI Formulas: Proportion

CI for Proportion p (large sample) — Procedure 3.4:

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \;\leq\; p \;\leq\; \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \tag{3.7}$$

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Sample Size for Proportion:

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If no prior $\hat{p}$ is available, use $\hat{p} = 0.5$ — the conservative version, (3.9) in the textbook.

t-Distribution Critical Values, Table 3.3 ($t_{\alpha,\nu}$ = value where $P(T > t_{\alpha,\nu}) = \alpha$)

df (ν)	Upper-tail probability α			
	0.10	**0.05**	**0.025**	**0.01**
1	3.078	6.314	12.706	31.821
2	1.886	2.920	4.303	6.965
3	1.638	2.353	3.182	4.541
5	1.476	2.015	2.571	3.365
10	1.372	1.812	2.228	2.764
15	1.341	1.753	2.131	2.602
20	1.325	1.725	2.086	2.528
30	1.310	1.697	2.042	2.457
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Example: For 95% two-sided CI with $n = 11$ samples (df = 10): Use $t_{0.025,10} = 2.228$

χ^2-Distribution Critical Values, Table 3.4 ($\chi^2_{\alpha,\nu}$ = value s.t. $P(\chi^2 > \chi^2_{\alpha,\nu}) = \alpha$)

	Upper-tail probability α			
df (ν)	**0.975**	**0.95**	**0.05**	**0.025**
1	0.001	0.004	3.841	5.024
2	0.051	0.103	5.991	7.378
5	0.831	1.145	11.070	12.833
9	2.700	3.325	16.919	19.023
10	3.247	3.940	18.307	20.483
19	8.907	10.117	30.144	32.852
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Example: For 95% CI for σ^2 with $n = 10$ ($df = 9$): Use $\chi^2_{0.975,9} = 2.700$ and $\chi^2_{0.025,9} = 19.023$

Summary

- When σ is **unknown**, use the *t*-**distribution** (3.5):
  - Replace σ with s
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- For **proportion** p, use **Normal approximation** (3.7):
  - Requires large sample: $n\hat{p} \geq 10$ and $n(1 - \hat{p}) \geq 10$
  - Standard error: $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Questions you might be asked about confidence intervals

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- Choose the appropriate distribution (z, t, or χ^2) for a given problem

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