ISE 315: Engineering Statistics

Lecture 17: Simple Linear Regression — A Full Worked Fit

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Based on Montgomery & Runger, Applied Statistics and Probability for Engineers, 6th Ed.

Lecture 17

Simple Linear Regression, Continued: A Full Worked Fit (Chapter 6)

Recap: The Model and the Estimators

From Lecture 15

The simple linear regression model (6.2):

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, \ldots, n$$

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Least squares building blocks and estimators:

Quantity	Formula	Book eq.
Spread of x	$S_{xx} = \sum x_i^2 - (\sum x_i)^2/n$	(6.7)
Co-movement	$S_{xy} = \sum x_i y_i - (\sum x_i)(\sum y_i)/n$	(6.8)
Slope	$\hat{\beta}_1 = S_{xy}/S_{xx}$	(6.9)
Intercept	$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$	(6.10)
[illegible]	$\hat{\sigma}^2 = SS_E/(n-2)$	(6.14)

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Recap: The AI Verification Fit

Quick Warm-Up

In Lecture 15 we fit the AI safety verification data (x = testing hours, y = failures per 1000 inspections, $n = 8$):

$$\hat{y} = 54.87 - 0.8452\,x, \qquad \hat{\sigma}^2 = 1.621$$

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$x = 32$ is **inside** the observed range $10 \le x \le 45$, so this is interpolation — a

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- Exam-style practice problem (startup pricing experiment)

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Eight steps, every one on the formula sheet. Let's do all of them once, carefully.

Worked Example: Truck Load vs. Fuel Consumption

Supply Chain Data

A fleet analyst wants to predict fuel consumption from payload so she can quote delivery costs. For $n = 8$ line-haul trips she records the load x (tonnes) and the fuel consumption y (L/100 km):

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y (L/100 km)	23.0	24.2	28.1	28.6	32.9	33.4	37.2	37.8

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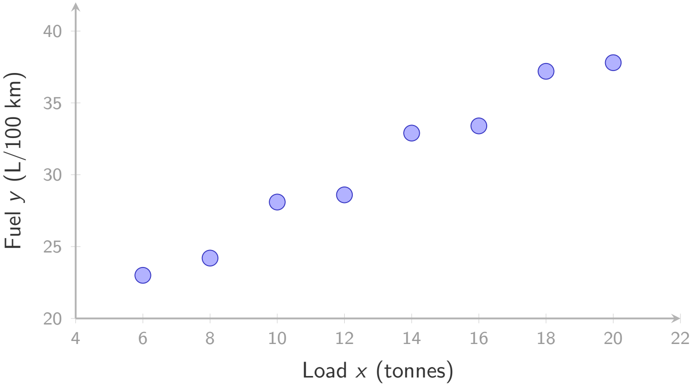
Questions we will answer:

- How much extra fuel does each additional tonne cost?
- How much scatter remains around the fitted line?

Textbook: Chapter 6, §6.3–§6.4 ((6.7)–(6.14))

Step 0: Always Plot First



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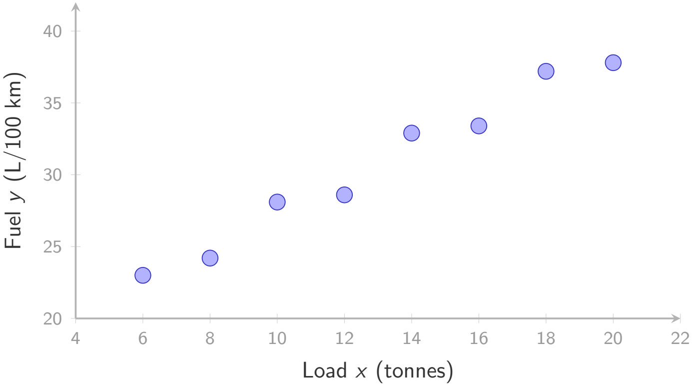

The scatter is clearly **linear and increasing**, with modest noise. A straight-line model (6.2) is reasonable. Now we fit it.

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$$\sum x_i^2 = 36 + 64 + \cdots + 400 = 1520$$

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Step 2a: The Spread of x

Using (6.7)

Write the formula first — equation (6.7) in the textbook:

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum x_i\right)^2}{n} \tag{6.7}$$

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$$S_{xx} = 1520 - \frac{(104)^2}{8} = 1520 - \frac{10816}{8}$$

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Step 2b: The Co-Movement of x and y

Using (6.8)

Again, formula first — equation (6.8):

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$S_{xy} > 0$: heavier loads move *with* higher fuel consumption. The slope will be positive — exactly what the scatter plot showed.

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For a cost estimate: on the 400-km Dammam–Riyadh run, one extra tonne means roughly $4 \times 1.124 \approx 4.5$ extra litres of diesel per trip.

Step 4: The Intercept and the Fitted Line

Using (6.10) and (6.3)

The intercept — equation (6.10):

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{6.10}$$

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Substitute ($\bar{x} = 13$, $\bar{y} = 30.65$, $\hat{\beta}_1 = 1.124$):

$$\hat{\beta}_0 = 30.65 - (1.124)(13) = 30.65 - 14.61 = 16.04$$

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$$\boxed{\hat{y} = 16.04 + 1.124\,x}$$

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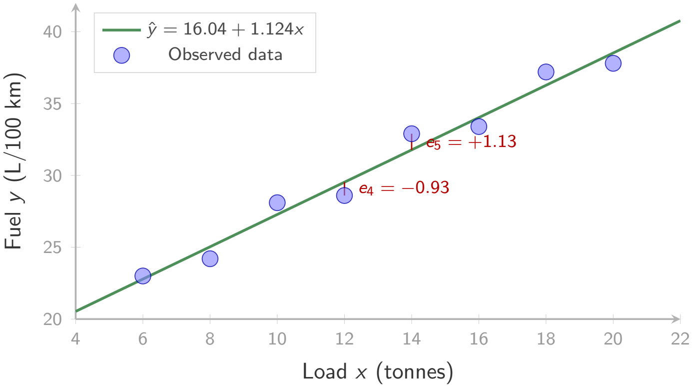
The intercept (≈ 16 L/100 km) is the model's estimate for an *empty* truck. Note

The Fitted Line Over the Data



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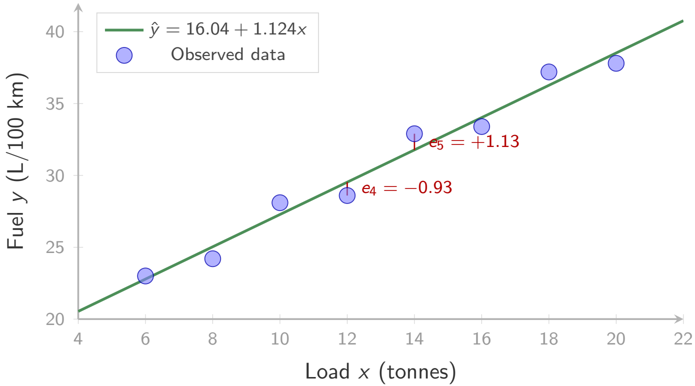

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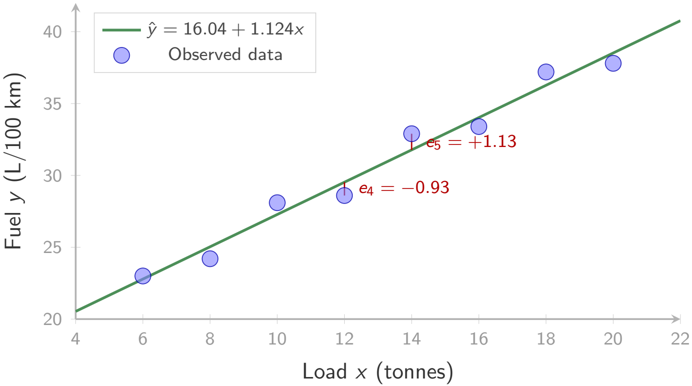

The vertical gaps between points and line are the **residuals** (6.4). We compute all eight next.

Step 6: The Residuals

Using (6.4): $e_i = y_i - \hat{y}_i$

For each trip, compute the fitted value $\hat{y}_i = 16.04 + 1.124\, x_i$, then the residual:

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i	x_i	y_i	$\hat{y}_i$	$e_i = y_i - \hat{y}_i$
1	6	23.0	22.78	$+0.22$
2	8	24.2	25.03	-0.83
3	10	28.1	27.28	$+0.82$
4	12	28.6	29.53	-0.93
5	14	32.9	31.77	$+1.13$
6	16	33.4	34.02	-0.62
7	18	37.2	36.27	$+0.93$
8	20	37.8	38.52	-0.72

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5	14	32.9	31.77	$+1.13$
6	16	33.4	34.02	-0.62
7	18	37.2	36.27	$+0.93$
8	20	37.8	38.52	-0.72

Built-in check: $\sum e_i = 0.22 - 0.83 + 0.82 - 0.93 + 1.13 - 0.62 + 0.93 - 0.72 = 0.00$ ✓

Step 7: The Error Sum of Squares, Two Ways

(6.11) vs. the Shortcut (6.13)

Way 1 — direct, from the residual table (6.11):

$$SS_E = \sum_{i=1}^{8} e_i^2 = 0.22^2 + 0.83^2 + \cdots + 0.72^2 = 5.30$$

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$$S_{yy} = \sum y_i^2 - \frac{(\sum y_i)^2}{n} = 7732.86 - \frac{(245.2)^2}{8} = 7732.86 - 7515.38 = 217.48$$

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Then (6.13):

$$SS_E = S_{yy} - \hat{\beta}_1 \, S_{xy} = 217.48 - (1.124)(188.8) = 217.48 - 212.18 = 5.30$$

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Then (6.13):

$$SS_E = S_{yy} - \hat{\beta}_1 S_{xy} = 217.48 - (1.124)(188.8) = 217.48 - 212.18 = 5.30$$

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The complete answer: $\hat{y} = 16.04 + 1.124\,x$ with $\hat{\sigma}^2 = 0.884$. These three numbers ($\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$) are the full output of Procedure 6.1 — and the inputs to the

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x (discount, %)	5	10	15	20	25
y (conversion, %)	3.1	4.0	4.4	5.3	5.7

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$\sum x$	$\sum y$	$\sum x^2$	$\sum y^2$	$\sum xy$
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Compute S_{xx} (6.7) and S_{xy} (6.8):

$$S_{xx} = \sum x_i^2 - \frac{(\sum x_i)^2}{n} = 1375 - \frac{(75)^2}{5} = 1375 - 1125 = 250$$

$$S_{xy} = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n} = 370 - \frac{(75)(22.5)}{5} = 370 - 337.5 = 32.5$$

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Practice Problem — Solution (Continued)

Parts (b), (c), and (d)

(b) Estimate conversion at $x = 20$:

$$\hat{y} = 2.55 + 0.13(20) = 2.55 + 2.60 = \boxed{5.15\%}$$

Practice Problem — Solution (Continued)

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$$e = y - \hat{y} = 5.3 - 5.15 = \boxed{+0.15}$$

Since $e > 0$ (actual $>$ predicted), the model **underestimates** the true value there.

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(d) Expected change for 8 additional discount points:

$$\Delta\hat{y} = \hat{\beta}_1 \times 8 = 0.13 \times 8 = \boxed{1.04 \text{ percentage points}}$$

Interpretation: the slope $\hat{\beta}_1 = 0.13$ means each extra point of discount is expected

Lecture 17 Summary

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- **Next lecture:** tests and confidence intervals on β_1 and β_0, and the regression ANOVA table.

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