ISE 315: Engineering Statistics

Lecture 24: Random Effects and Randomized Block Designs

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Based on Montgomery & Runger, Applied Statistics and Probability for Engineers, 6th Ed.

Lecture 24

Random-Effects Model and the Randomized Complete Block Design
(§8.10, §8.11)

Lecture 24 Outline

- Fixed vs. random factors — a question of inference scope

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- The random-effects model and variance components (8.20)

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- Fully worked RCBD: four routing policies blocked by day (Procedure 8.5)

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- The RCBD: model, decomposition (8.22), F-test (8.24)
- Fully worked RCBD: four routing policies blocked by day (Procedure 8.5)
- RCBD post-hoc comparisons and residual checks

Fixed vs. Random Factors

§8.10

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Decided by the design, not the data: picked the levels on purpose $\Rightarrow$ fixed; sampled them and want to speak about the population $\Rightarrow$ random.

The Random-Effects Model

The equation looks identical to (8.2):

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

but now $\tau_i \sim N(0, \sigma_\tau^2)$ is *random*, independent of $\varepsilon_{ij} \sim N(0, \sigma^2)$.

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Variance of any observation splits into two **variance components**:

$$V(Y_{ij}) = \sigma_\tau^2 + \sigma^2 \tag{8.18}$$

between-treatment σ_τ^2 and within-treatment σ^2.

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between-treatment σ_τ^2 and within-treatment σ^2.
The hypotheses change: individual τ_i are no longer parameters to compare —

$$H_0 : \sigma_\tau^2 = 0 \qquad \text{vs.} \qquad H_1 : \sigma_\tau^2 > 0$$

Does the factor contribute *any* variability to the population?

Estimating Variance Components

Procedure 8.4

The pleasant surprise: the arithmetic doesn't change. Same SS, same df, same $F_0 = MS_{\mathrm{Trt}}/MS_E$, same critical value $f_{\alpha,\, a-1,\, N-a}$ (Procedure 8.1).

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What changes is the interpretation, via the expected mean squares — compare with (8.10):

$$E[MS_E] = \sigma^2, \qquad E[MS_{\text{Treatments}}] = \sigma^2 + n\,\sigma_\tau^2 \tag{8.19}$$

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Solve two equations for two unknowns — the **method-of-moments estimators**:

$$\hat{\sigma}^2 = MS_E, \qquad \hat{\sigma}_\tau^2 = \frac{MS_{\text{Treatments}} - MS_E}{n} \tag{8.20}$$

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Caveat: if $MS_{\text{Trt}} < MS_E$, (8.20) goes negative — impossible for a variance. Report $\hat{\sigma}_\tau^2 = 0$: no detectable contribution. Useful summary: the proportion $\hat{\sigma}_\tau^2/(\hat{\sigma}_\tau^2 + \hat{\sigma}^2)$.

Worked Example: Warehouse-to-Warehouse Variability

Table 8.12

A logistics network samples $a = 4$ warehouses *at random* from its 30 sites and times $n = 5$ random orders at each (minutes). How much of picking-time variability is a site effect? Following Procedure 8.1:

Source	SS	df	MS	F_0
Warehouses	126.0	3	42.00	7.00
Error	96.0	16	6.000	
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Step 1 — the test. $H_0 : \sigma_\tau^2 = 0$; $f_{0.05,\,3,\,16} = 3.24$. Since $F_0 = 7.00 > 3.24$, **reject** ($p \approx 0.003$): warehouse differences are a real source of variability across the network.

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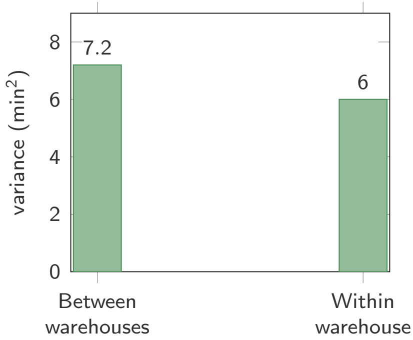
Step 2 — components (8.20):

$$\frac{42.00 - 6.000}{\cdots} \quad \frac{36.00}{\cdots}$$

Interpreting the Split

Figure 8.7



Interpreting the Split

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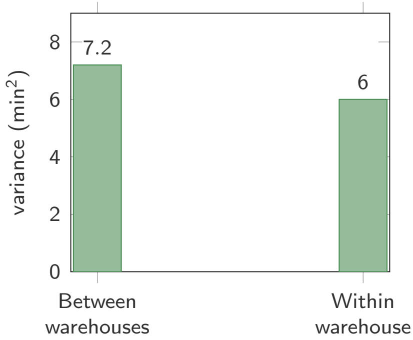

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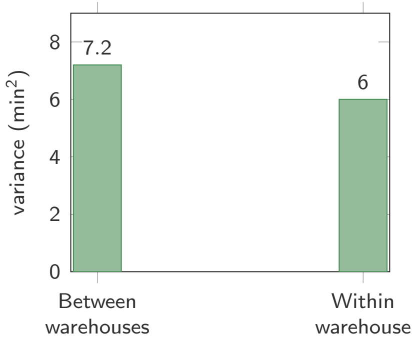

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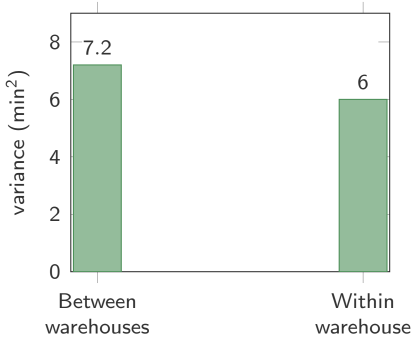

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More than half of picking-time variability is a site effect. Standardizing processes *across* warehouses attacks the larger component; training pickers *within* a site attacks the smaller one.

Quick Check: Variance Components

AI safety

Q. An autonomous-vehicle operator randomly selects 5 vehicles from its fleet and measures a perception-latency score on $n = 6$ randomized runs per vehicle. ANOVA gives $MS_{\text{Treatments}} = 12.0$, $MS_E = 4.0$. The estimated between-vehicle component $\hat{\sigma}^2_\tau$ equals:

(A) $12.0 - 4.0 = 8.0$

(B) $(12.0 - 4.0)/6 = 1.33$

(C) $12.0/4.0 = 3.0$

(D) 4.0, since $\hat{\sigma}^2 = MS_E$

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Answer: (B). By (8.20), $\hat{\sigma}_\tau^2 = (MS_{\text{Trt}} - MS_E)/n = 8/6 = 1.33$. The divisor is n because n multiplies σ_τ^2 in (8.19). (A) forgets the division; (C) is F_0, not a component; (D) is the *within*-vehicle component. Proportion between vehicles: $1.33/5.33 = 25\%$.

The Need for Blocking

§8.11

Scenario (back to fixed effects). A last-mile logistics team compares 4 routing policies for its delivery vans. Traffic varies strongly by **day of week** — a variable that affects delivery time but is not the question. That is a **nuisance factor**.

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Startup view: weekly signup cohorts are blocks. Testing onboarding flows *within* each cohort keeps a lucky week from crowning the wrong variant.

The Randomized Complete Block Design

Layout: a treatments, b blocks; every treatment appears *exactly once* in every block ($N = ab$), in random order *within* each block. "Complete" $=$ every block contains every treatment.

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The model adds one term to (8.2):

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, \qquad i = 1, \ldots, a, \quad j = 1, \ldots, b \tag{8.21}$$

- τ_i: treatment effect ($\sum_i \tau_i = 0$); β_j: block effect ($\sum_j \beta_j = 0$)
- $\varepsilon_{ij} \sim N(0, \sigma^2)$ independent; the model is *additive*

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Hypotheses — treatments only:

$$H_0 : \tau_1 = \cdots = \tau_a = 0 \quad \text{vs.} \quad H_1 : \text{at least one } \tau_i \neq 0$$

Blocks are not on trial; they are there to be subtracted.

RCBD: Decomposition and ANOVA Table

Table 8.13

Total variability now partitions into *three* pieces:

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks}} + SS_E \tag{8.22}$$

with computing forms (8.23) using row totals $y_{i\cdot}$, column totals $y_{\cdot j}$, and $y_{\cdot\cdot}^2/ab$.

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Treatment test (8.24): reject H_0 if $F_0 = MS_{\text{Trt}}/MS_E > f_{\alpha,\, a-1,\, (a-1)(b-1)}$.

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Procedure 8.5: RCBD Analysis

The road map for the worked example

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4. Compute the mean squares and F_0 (8.24); reject if $F_0 > f_{\alpha,\, a-1,\, (a-1)(b-1)}$.

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5. If rejected: multiple comparisons with (8.25) or (8.26) — note b in place of n.

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5. If rejected: multiple comparisons with (8.25) or (8.26) — note b in place of n.
6. Check residuals (8.27), including within-block patterns (interaction warning).

Worked RCBD: Four Routing Policies, Blocked by Day

Table 8.14

Policies: **P1** current software, **P2** new optimization engine, **P3** driver-chosen routes, **P4** optimizer with live traffic disabled. Response: mean delivery time per package (minutes). Each policy runs once on each working day, order randomized within day: $a = 4$, $b = 5$, $N = 20$.

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P1. Current	41	40	42	43	44	210	42.0
P2. Optimizer	37	39	41	41	42	200	40.0
P3. Driver-chosen	43	45	45	45	47	225	45.0
P4. Optimizer, no live	39	40	40	43	43	205	41.0
$y_{\cdot j}$	160	164	168	172	176	840	
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Step 1 — totals done. Block means climb steadily from 40 (Sun) to 44 (Thu); the day

Worked RCBD: Step 2, Sums of Squares

Using (8.23)

Total: $\sum\sum y_{ij}^2 = 35398$ and $y_{..}^2/ab = 840^2/20 = 35280$, so $SS_T = 35398 - 35280 = 118.0$

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Treatments (divide by $b = 5$):

$$SS_{\text{Trt}} = \frac{210^2 + 200^2 + 225^2 + 205^2}{5} - 35280 = \frac{176450}{5} - 35280 = 70.00$$

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Blocks (divide by $a = 4$):

$$SS_{\text{Blk}} = \frac{160^2 + 164^2 + 168^2 + 172^2 + 176^2}{4} - 35280 = \frac{141280}{4} - 35280 = 40.00$$

Worked RCBD: Step 2, Sums of Squares

Using (8.23)

Total: $\sum\sum y_{ij}^2 = 35398$ and $y_{..}^2/ab = 840^2/20 = 35280$, so $SS_T = 35398 - 35280 = 118.0$

Treatments (divide by $b = 5$):

$$SS_{\text{Trt}} = \frac{210^2 + 200^2 + 225^2 + 205^2}{5} - 35280 = \frac{176450}{5} - 35280 = 70.00$$

Blocks (divide by $a = 4$):

$$SS_{\text{Blk}} = \frac{160^2 + 164^2 + 168^2 + 172^2 + 176^2}{4} - 35280 = \frac{141280}{4} - 35280 = 40.00$$

Error by subtraction (8.22): $SS_E = 118 - 70 - 40 = 8.000$

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Check with definition forms: $SS_{\text{Trt}} = 5[0^2 + (-2)^2 + 3^2 + (-1)^2] = 5(14) = 70$ ✓;

Worked RCBD: Steps 3–4, Table and Decision

Table 8.15

Step 3 — df: $a - 1 = 3, \quad b - 1 = 4, \quad (a-1)(b-1) = 12, \quad$ total 19 $(3 + 4 + 12 = 19\ \checkmark)$

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Source	**SS**	**df**	**MS**	F_0
Policies (treatments)	70.00	3	23.33	35.00
Days (blocks)	40.00	4	10.00	
Error	8.000	12	0.6667	
Total	118.0	19		

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The same data, analyzed two ways

Suppose the team had ignored days and analyzed the same 20 numbers as a completely randomized design. The day variability folds into the error row:

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Still significant here — but a smaller policy effect would have been lost entirely. Blocking pays whenever the nuisance variability it removes outweighs the $b - 1$ error df it costs, which is almost always when the blocking variable matters.

Multiple Comparisons in the RCBD

Step 5 of Procedure 8.5

Same Fisher/Tukey logic as Lecture 23, with **two mechanical substitutions**:

- each treatment mean averages b observations (one per block) $\Rightarrow$ b replaces n
- error df are now $(a-1)(b-1)$

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Tukey HSD for RCBD (8.26):

$$T_\alpha = q_\alpha\big(a,\ (a-1)(b-1)\big)\sqrt{\frac{MS_E}{b}}$$

Declare $\bar{y}_{i\cdot}$ and $\bar{y}_{j\cdot}$ different when $|\bar{y}_{i\cdot} - \bar{y}_{j\cdot}|$ exceeds the yardstick.

Worked RCBD: Tukey Post-Hoc

Which policies differ?

Ingredients: $a = 4$, $df_E = 12$, $MS_E = 0.6667$, $b = 5$; $q_{0.05}(4, 12) = 4.20$. By (8.26):

$$T_{0.05} = 4.20\sqrt{\frac{0.6667}{5}} = 4.20\sqrt{0.1333} = 4.20 \times 0.3651 = 1.534$$

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Ordered means: P2 (40.0), P4 (41.0), P1 (42.0), P3 (45.0). All six gaps:

Pair	gap	> 1.534?	Pair	gap	> 1.534?
P2–P4	1.0	No	P2–P3	5.0	Yes
P4–P1	1.0	No	P4–P3	4.0	Yes
P2–P1	2.0	Yes	P1–P3	3.0	Yes

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RCBD Residuals and Model Checking

Step 6 of Procedure 8.5

The fitted value under the additive model combines treatment and block means, $\hat{y}_{ij} = \bar{y}_{i\cdot} + \bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}$, so

$$e_{ij} = y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot} \tag{8.27}$$

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Usual plots (normal plot, residuals vs. fitted) **plus one specific to the RCBD:** look for a systematic pattern of residuals *within blocks* — e.g., one treatment always high early in the week and low late.

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Usual plots (normal plot, residuals vs. fitted) **plus one specific to the RCBD:** look for a systematic pattern of residuals *within blocks* — e.g., one treatment always high early in the week and low late.

Such a pattern signals a **treatment-by-block interaction**, violating the additivity of (8.21). The factorial designs of Chapter 9 are built to model it.

Worked RCBD: The Residual Matrix

Table 8.16

Example cell — P1 on Sunday: $\hat{y}_{11} = 42.0 + 40.0 - 42.0 = 40$, so $e_{11} = 41 - 40 = +1$. Repeating (8.27) for all 20 cells:

Worked RCBD: The Residual Matrix

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Residual e_{ij}	Sun	Mon	Tue	Wed	Thu	Row sum
P1	$+1$	-1	0	0	0	0
P2	-1	0	$+1$	0	0	0
P3	0	$+1$	0	-1	0	0
P4	0	0	-1	$+1$	0	0
Column sum	0	0	0	0	0	0

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P4	0	0	-1	$+1$	0	0
Column sum	0	0	0	0	0	0

Checks: every row and column sums to zero (automatic — a free arithmetic check); all residuals are 0 or ± 1; no policy is systematically high early and low late. The additive model is adequate, and the squared entries confirm $SS_E = 8$.

Quick Check: RCBD Degrees of Freedom

Startup

Q. A startup tests $a = 4$ pricing-page variants in an RCBD blocked on $b = 5$ acquisition channels (every variant shown within every channel). The error degrees of freedom for the treatment F-test are:

(A) $N - 1 = 19$

(B) $N - a = 16$

(C) $(a - 1)(b - 1) = 12$

(D) $a(n - 1) = 16$

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(D) $a(n - 1) = 16$

Answer: (C). In the RCBD, $df_E = (a-1)(b-1) = 3\times 4 = 12$, as in (8.24). (A) is the total df; (B) is the one-way error df, which over-counts by the $b - 1 = 4$ block df; (D) confuses the CRD replicate count n with the block count b — each treatment appears once per block here.

Quick Check: Why Block?

Q. An AI evaluation team compares scoring rubrics, blocking on annotator. The treatment F_0 is much larger in the blocked analysis than in a one-way analysis of the same data. The most accurate explanation:

(A) Blocking always increases $SS_{\text{Treatments}}$.

(B) Blocking removes annotator variability from SS_E, shrinking MS_E and raising F_0.

(C) Blocking changes the treatment means.

(D) Blocking adds more degrees of freedom to the error term.

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Answer: (B). By (8.22), SS_{Blocks} is carved out of what a one-way analysis would call error, so MS_E shrinks and $F_0 = MS_{\text{Trt}}/MS_E$ rises. (A) is false — SS_{Trt} is unchanged; (C) confuses the model with the means; (D) is backwards — blocking *decreases* df_E.

Choosing the Right Design

Table 8.17

Situation	Design and analysis	Inference target
Specific levels chosen on purpose	Fixed-effects one-way ANOVA (Procedure 8.1)	Compare those means
Levels sampled from a population	Random-effects model (Procedure 8.4)	Variance components σ_τ^2, σ^2
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The decision is made when the experiment is planned, before any data exist:

- Were the levels chosen on purpose, or sampled?
- What nuisance sources can I anticipate — and block?

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